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Positive solutions of a singular nonlinear third order two-point boundary value problem ☆

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Abstract

In this paper some existence results of positive solutions for the following singular nonlinear third order two-point boundary value problem:

$$x'''(t) - \alpha(t) f\big(t, x(t)\big) = 0, \quad a < t < b,$$
$$x(a) = x(b) = x''(b) = 0$$

are established, where $\alpha \in C((a,b),[0,+\infty))$, $f \in C([a,b] \times (0,+\infty),[0,+\infty))$, $\alpha(t)$ may be singular at $t = a, b$ and $f(t,x)$ may be singular at $x = 0$.

Keywords: Singular nonlinear third order two-point boundary value problem; Positive solution; Green's function; Krasnosel'skii fixed-point theorem

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1. Introduction

In recent years, several authors investigated the existence problems of solutions, periodic solutions, positive solutions, multiple solutions and numerical solutions for the following nonlinear three order differential equations:

$$x'''(t) - f\big(x(t)\big) = 0, \quad 0 < t < +\infty, \tag{1.1}$$

$$x'''(t) - f\big(t, x(t)\big) = 0, \quad a \leqslant t \leqslant b, \tag{1.2}$$

$$x'''(t) - \lambda a(t) f\big(t, x(t)\big) = 0, \quad 0 < t < 1, \tag{1.3}$$

under various boundary conditions [1–9,11–15]. Khan and Aziz [8] considered the numerical solutions for Eq. (1.2) under two-point boundary condition $x(a) = k_1$, $x'(a) = k_2$, $x(b) = k_3$. Cabada [5] and Yao and Feng [15] used lower and upper solutions method and fixed point theorems to establish the existence of periodic solutions and positive solutions for Eq. (1.2) under two-point boundary conditions $u^{(i)}(0) = u^{(i)}(2\pi)$, $i = 0, 1, 2$ with $a = 0$, $b = 2\pi$, and $x(0) = x'(0) = x'(1) = 0$ with $a = 0$, $b = 1$, respectively. By means of the Krasnosel'skii, Leggett–Williams and five functionals fixed-point theorems, Anderson [2], Anderson and Davis [3] and Yao [14] established some sufficient conditions, which guarantee the existence of multiple positive solutions to Eq. (1.2) under three-point boundary conditions $x(t_1) = x'(t_2) = 0$, $\gamma x(t_3) + \delta x''(t_3) = 0$, or $x(t_1) = x'(t_2) = x''(t_3) = 0$ with $a = t_1$, $b = t_3$, or $x(0) = x'(\eta) = x''(1) = 0$ with $a = 0$, $b = 1$, respectively. Wang and Zhang [13] and Jiang and Agarwal [7] studied the existence and uniqueness of solutions for the singular third order differential equation (1.1) with $f(s) = (1-s)^{\lambda} g(s)$ under boundary condition $x(0) = 0, x(+\infty) = 1$, $x'(+\infty) = x''(+\infty) = 0$. Recently, Sun [11] discussed the existence of multiple positive solutions for the singular nonlinear third order differential equation (1.3) under three-point boundary condition $x(0) = x'(\eta) = x''(1) = 0$, where the function $a(t)$ may be singular at $t = 0, 1$, however the function f has no singularity. To the best of the authors' knowledge there is no literature referred to the existence of positive solutions for the following singular nonlinear third order two-point boundary value problem:

$$x'''(t) - \alpha(t) f\big(t, x(t)\big) = 0, \quad a < t < b, \tag{1.4}$$

$$x(a) = x(b) = x''(b) = 0, \tag{1.5}$$

where $\alpha \in C((a, b), [0, +\infty))$, $f \in C([a, b] \times (0, +\infty), [0, +\infty))$, $\alpha(t)$ may be singular at $t = a, b$ and $f(t, x)$ may be singular at $x = 0$. The purpose of this paper is to fill in the gap in this area. Applying the positivity of the Green's function $G(t, s)$ and the Krasnosel'skii fixed-point theorem, we give a few sufficient conditions of the existence of positive solutions for Eqs. (1.4), (1.5) under certain conditions.

This paper is organized as follows. In Section 2 several notation and lemmas are introduced. A few sufficient conditions of the existence of positive solutions for Eqs. (1.4), (1.5) are discussed in Section 3. As applications of our results, two examples are presented in Section 4.

2. Preliminaries

Let X be a Banach space and Y be a cone in X. A function x is said to be a *positive solution* of Eqs. (1.4), (1.5) if x is a solution of Eqs. (1.4), (1.5) and $x(t) > 0$ for each $t \in (a, b)$. Throughout this paper, we assume that $C[a, b]$ denotes the Banach space of all continuous functions on $[a, b]$ with the supremum norm $\|u\| =: \sup_{t \in [a,b]} |u(t)|$ for each $u \in C[a, b]$, p and q are constants with

$a < p < q < b$, $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ are strictly decreasing and strictly increasing sequences, respectively, with $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $a_1 < \frac{a+b}{2} < b_1$,

$$D_n = \min\left\{\frac{a_n - a}{b-a}, \frac{b-b_n}{b-a}\right\}, \quad A_n = [a, a_n] \cup [b_n, b], \quad n \geqslant 1,$$

$$P_1 = \big\{x \in C[a,b]\colon\ x \text{ is nonnegative and concave on } [a,b]\big\},$$

$$P = \big\{x \in P_1\colon\ x(t) \geqslant h(t)\|x\|,\ t \in [p,q]\big\},$$

$$P_r = \big\{x \in P\colon\ \|x\| < r\big\}, \quad \partial P_r = \big\{x \in P\colon\ \|x\| = r\big\}, \quad r > 0,$$

$$P_{r,s} = \big\{x \in P\colon\ r \leqslant \|x\| \leqslant s\big\}, \quad s > r > 0,$$

$$h(t) = \frac{(b-t)(t-a)}{2(b-a)^2}, \quad l(t) = \min\left\{\frac{t-a}{b-a}, \frac{b-t}{b-a}\right\}, \quad g(t) = (t-a)^2, \quad t \in [a,b],$$

and

$$G(t,s) = \begin{cases} \frac{(b-t)(s-a)^2}{2(b-a)}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{(t-a)[(b-a)(b-t)-(b-s)^2]}{2(b-a)}, & a \leqslant t < s \leqslant b, \end{cases}$$

is the Green's function of the homogeneous problem $x'''(t) = 0$ satisfying the boundary condition (1.5). It is easy to verify that P_1 and P are cones of $C[a,b]$. Moreover, we use also the following assumptions:

(C_1) $\alpha : (a,b) \to [0,+\infty)$ is continuous and

$$\int_p^q g(t)\alpha(t)\,dt > 0 \quad \text{and} \quad \int_a^b g(t)\alpha(t)\,dt < +\infty,$$

and

(C_2) $f : [a,b] \times (0,+\infty) \to [0,+\infty)$ is continuous and

$$\lim_{n\to+\infty} \sup_{x\in P_{c,d}} \int_{A_n} g(t)\alpha(t) f\big(t, x(t)\big)\,dt = 0 \quad \text{for all } c, d \text{ with } 0 < c < d.$$

Lemma 2.1.

(a) *If* $x \in P_1$, *then* $x(t) \geqslant \|x\| l(t)$, $t \in [a,b]$;
(b) *If* $x, y \in P_1$ *and* $z(t) = \min\{x(t), y(t)\}$, $t \in [a,b]$, *then* $z \in P_1$.

Proof. Let $x \in P_1$. If $x = \theta$, clearly, $x(t) = 0 = \|x\| l(t)$, $t \in [a,b]$; if $x \in P_1 \setminus \{\theta\}$, it follows that $\|x\| = x(v) > 0$ for some $v \in (a,b)$ and

$$\frac{x(t) - x(a)}{t-a} \geqslant \frac{x(v) - x(t)}{v-t}, \quad t \in (a,v),$$

and

$$\frac{x(t) - x(v)}{t-v} \geqslant \frac{x(b) - x(t)}{b-t}, \quad t \in (v,b),$$

which imply that

$$x(t) \geqslant \frac{v-t}{v-a}x(a) + \frac{t-a}{v-a}x(v) \geqslant \frac{t-a}{b-a}x(v) \geqslant \|x\|l(t), \quad t \in (a,v), \tag{2.1}$$

and

$$x(t) \geqslant \frac{b-t}{b-v}x(v) + \frac{t-v}{b-v}x(b) \geqslant \frac{b-t}{b-a}x(v) \geqslant \|x\|l(t), \quad t \in (v,b). \tag{2.2}$$

It follows from (2.1), (2.2) and the continuity of x and l that $x(t) \geqslant \|x\|l(t)$ for $t \in [a,b]$, that is, (a) holds. In order to prove (b), we need only verify the concavity of z on $[a,b]$. In fact, for any $t,s \in [a,b]$ and $c \in [0,1]$, we get that

$$\begin{aligned} z\bigl(ct+(1-c)s\bigr) &= \min\bigl\{x\bigl(ct+(1-c)s\bigr), y\bigl(ct+(1-c)s\bigr)\bigr\} \\ &\geqslant \min\bigl\{cx(t)+(1-c)x(s), cy(t)+(1-c)y(s)\bigr\} \\ &\geqslant c\min\bigl\{x(t),y(t)\bigr\} + (1-c)\min\bigl\{x(s),y(s)\bigr\} \\ &= cz(t)+(1-c)z(s). \end{aligned}$$

Hence (b) holds. This completes the proof. □

Lemma 2.2.

(a) $0 \leqslant 2h(t) \leqslant l(t) \leqslant 1$, $t \in [a,b]$;
(b) $0 \leqslant h(t)g(s) \leqslant G(t,s) \leqslant g(s)$, $t,s \in [a,b]$;
(c) *For each* $s \in [a,b]$, $G(\cdot,s)$ *is concave in the first argument on* $[a,b]$.

Proof. It is clear that (a) holds. Now we show that (b) holds. Let t,s be in $[a,b]$. If $s \leqslant t$, it follows that

$$0 \leqslant h(t)g(s) \leqslant \frac{b-t}{2(b-a)}g(s) = G(t,s) = \frac{(b-t)(s-a)^2}{2(b-a)} \leqslant g(s);$$

if $t < s$, we conclude easily that

$$G(t,s) = \frac{(t-a)[(b-a)(b-t)-(b-s)^2]}{2(b-a)} \leqslant \frac{(s-a)[(b-a)^2-(b-s)^2]}{2(b-a)} \leqslant g(s)$$

and

$$\begin{aligned} G(t,s) = \frac{(t-a)[(b-s)(s-t)+(s-a)(b-t)]}{2(b-a)} &\geqslant \frac{(t-a)(s-a)(b-t)}{2(b-a)} \geqslant h(t)g(s) \\ &\geqslant 0. \end{aligned}$$

That is, (b) holds. Let t,r,s be in $[a,b]$ with $t \leqslant r$ and c be in $[0,1]$. In order to show (c), we have to consider the following cases:

Case 1. Suppose that $r \leqslant s$. Since $ct+(1-c)r \leqslant s$, it follows that

$$\begin{aligned} &G\bigl(ct+(1-c)r,s\bigr) - cG(t,s) - (1-c)G(r,s) \\ &\quad = \frac{1}{2(b-a)}\bigl\{\bigl[ct+(1-c)r-a\bigr]\bigl[(b-a)\bigl(b-ct-(1-c)r\bigr)-(b-s)^2\bigr] \\ &\qquad - c(t-a)\bigl[(b-a)(b-t)-(b-s)^2\bigr] - (1-c)(r-a)\bigl[(b-a)(b-r)-(b-s)^2\bigr]\bigr\} \\ &\quad = \frac{1}{2}c(1-c)(r-t)^2 \geqslant 0. \end{aligned}$$

Case 2. Suppose that $t \geqslant s$. Notice that $ct + (1-c)r \geqslant s$. It is easy to see that

$$\begin{aligned}
&G\big(ct + (1-c)r, s\big) - cG(t,s) - (1-c)G(r,s) \\
&\quad = \frac{(s-a)^2}{2(b-a)}\big[b - ct - (1-c)r - c(b-t) - (1-c)(b-r)\big] = 0.
\end{aligned}$$

Case 3. Suppose that $t < s < r$ and $ct + (1-c)r < s$. It is clear that

$$\begin{aligned}
&G\big(ct + (1-c)r, s\big) - cG(t,s) - (1-c)G(r,s) \\
&\quad = \frac{1}{2(b-a)}\big\{\big[ct + (1-c)r - a\big]\big[(b-a)\big(b - ct - (1-c)r\big) - (b-s)^2\big] \\
&\qquad - c(t-a)\big[(b-a)(b-t) - (b-s)^2\big] - (1-c)(b-r)(s-a)^2\big\} \\
&\quad = \frac{1-c}{2}\big[c(r-t)^2 - (r-s)^2\big] \\
&\quad \geqslant \frac{1-c}{2}\big[c^2(r-t)^2 - (r-s)^2\big] \\
&\quad = \frac{1-c}{2}\big[s - ct - (1-c)r\big]\big[r - s + c(r-t)\big] \geqslant 0.
\end{aligned}$$

Case 4. Suppose that $t < s < r$ and $ct + (1-c)r \geqslant s$. It follows that

$$\begin{aligned}
&G\big(ct + (1-c)r, s\big) - cG(t,s) - (1-c)G(r,s) \\
&\quad = \frac{1}{2(b-a)}\big\{\big[b - ct - (1-c)r\big](s-a)^2 - c(t-a)\big[(b-a)(b-t) - (b-s)^2\big] \\
&\qquad - (1-c)(b-r)(s-a)^2\big\} \\
&\quad = \frac{c}{2}(s-t)^2 \geqslant 0.
\end{aligned}$$

Hence (c) holds. This completes the proof. □

Lemma 2.3 *(Krasnosel'skii fixed-point theorem [10]). Let $(X, \|\cdot\|)$ be a Banach space and let $Y \subset X$ be a cone in X. Assume that A and B are open subsets of X with $\theta \in A$, $\overline{A} \subset B$ and $T : Y \cap (\overline{B} \setminus A) \to Y$ is a completely continuous operator such that, either*

(a) $\|Tu\| \leqslant \|u\|$ *for* $u \in Y \cap \partial A$, *and* $\|Tu\| \geqslant \|u\|$ *for* $u \in Y \cap \partial B$, *or*
(b) $\|Tu\| \geqslant \|u\|$ *for* $u \in Y \cap \partial A$, *and* $\|Tu\| \leqslant \|u\|$ *for* $u \in Y \cap \partial B$.

Then T has at least one fixed point in $Y \cap (\overline{B} \setminus A)$.

3. Existence of positive solutions

In this section, let

$$k = \left[\int_a^b g(t)\alpha(t)\,dt\right]^{-1}, \qquad m = \left[\min\big\{h(p), h(q)\big\}\int_p^q g(t)\alpha(t)\,dt\right]^{-1},$$

$$M(s) = \sup_{x \in \partial P_s} \int_a^b g(t)\alpha(t) f\big(t, x(t)\big)\, dt, \quad s > 0.$$

Now we are ready to establish a few sufficient conditions for the existence of positive solutions of Eqs. (1.4), (1.5) under certain conditions by applying the positivity of the Green's function $G(t,s)$ and the Krasnosel'skii fixed-point theorem.

Theorem 3.1. *Assume that there exist three positive constants β, j and r with $\beta > j$ and $r < k$ satisfying*

$$f(t,s) \geqslant jm, \quad (t,s) \in [p,q] \times \big[2j \min\big\{h(p), h(q)\big\}, j\big], \tag{3.1}$$

$$f(t,s) \leqslant rs, \quad (t,s) \in [a,b] \times [\beta, i], \tag{3.2}$$

where $i = \max\{\frac{\beta}{2\min\{h(p),h(q)\}}, \frac{kM(\beta)}{k-r}\}$. If (C$_1$) *and* (C$_2$) *hold, then Eqs.* (1.4), (1.5) *possess a positive solution in P.*

Proof. Let c and d be any positive constants with $c < d$. Define the operator $T : P_{c,d} \to P$ by

$$Tx(t) = \int_a^b G(t,s)\alpha(s) f\big(s, x(s)\big)\, ds, \quad (t,x) \in [a,b] \times P_{c,d}. \tag{3.3}$$

First of all we show that

$$\sup_{x \in P_{c,d}} \int_a^b g(t)\alpha(t) f\big(t, x(t)\big)\, dt < +\infty. \tag{3.4}$$

In fact, (C$_2$) yields that there exists some positive integer n satisfying

$$\sup_{x \in P_{c,d}} \int_{A_n} g(t)\alpha(t) f\big(t, x(t)\big)\, dt < 1. \tag{3.5}$$

It follows from Lemma 2.1 that for each $x \in P_{c,d}$ and $t \in [a_n, b_n]$

$$d \geqslant x(t) \geqslant \|x\| l(t) \geqslant cl(t) \geqslant c \min\big\{l(a_n), l(b_n)\big\} = cD_n. \tag{3.6}$$

Put $B = \max\{f(t,s):\ a \leqslant t \leqslant b,\ cD_n \leqslant s \leqslant d\}$. In view of (3.5) and (3.6), we deduce that

$$\begin{aligned}
&\sup_{x \in P_{c,d}} \int_a^b g(t)\alpha(t) f\big(t, x(t)\big)\, dt \\
&\quad \leqslant \sup_{x \in P_{c,d}} \int_{A_n} g(t)\alpha(t) f\big(t, x(t)\big)\, dt + \sup_{x \in P_{c,d}} \int_{[a,b]\setminus A_n} g(t)\alpha(t) f\big(t, x(t)\big)\, dt \\
&\quad \leqslant 1 + B \int_a^b g(t)\alpha(t)\, dt < +\infty,
\end{aligned}$$

which means that (3.4) holds. (C$_1$) and (C$_2$) ensure that T is well defined. Let $x \in P_{c,d}$. It follows from Lemmas 2.1 and 2.2 that Tx is nonnegative, concave and

$$
\begin{aligned}
Tx(t) &= \int_a^b G(t,s)\alpha(s) f\big(s, x(s)\big)\, ds \\
&\geqslant \int_a^b h(t) g(s)\alpha(s) f\big(s, x(s)\big)\, ds \geqslant h(t)\|Tx\|, \quad t \in [p,q],
\end{aligned}
$$

that is, $T : P_{c,d} \to P$.

Now we claim that $T : P_{c,d} \to P$ is completely continuous. As in the proof of (3.5) and (3.6), for any $x \in P_{c,d}$ we deduce that by (3.3), (C_1), (C_2) and Lemma 2.2

$$
\begin{aligned}
\|Tx\| &= \sup_{t\in[a,b]} \int_a^b G(t,s)\alpha(s) f\big(s, x(s)\big)\, ds \leqslant \int_a^b g(s)\alpha(s) f\big(s, x(s)\big)\, ds \\
&= \int_{A_n} g(s)\alpha(s) f\big(s, x(s)\big)\, ds + \int_{[a,b]\setminus A_n} g(s)\alpha(s) f\big(s, x(s)\big)\, ds \\
&\leqslant \sup_{u\in P_{c,d}} \int_{A_n} g(s)\alpha(s) f\big(s, u(s)\big)\, ds + B \int_{[a,b]\setminus A_n} g(s)\alpha(s)\, ds \\
&\leqslant 1 + B \int_a^b g(s)\alpha(s)\, ds,
\end{aligned}
$$

that is, the operator T is bounded on $P_{c,d}$. Notice that (C_2) implies that for given $\varepsilon > 0$, there exists some positive integer $n > 3$ such that

$$
\sup_{x\in P_{c,d}} \int_{A_n} g(s)\alpha(s) f\big(s, x(s)\big)\, ds < \frac{\varepsilon}{4}. \tag{3.7}
$$

It follows from the continuity of the Green's function G on $[a,b] \times [a,b]$ that there exists some constant $\delta > 0$ satisfying

$$
\big|G(t,s) - G(r,s)\big| < \frac{\varepsilon}{2(b-a)[1 + \max_{w\in[a_n,b_n]} \alpha(w) \cdot \max_{(v,w)\in[a_n,b_n]\times[cD_n,d]} f(v,w)]} \tag{3.8}
$$

for all $t, r, s \in [0,1]$ with $|t - r| < \delta$. In light of (3.3), (3.7) and (3.8), we deduce that

$$
\begin{aligned}
&\big|Tx(t) - Tx(r)\big| \\
&\quad \leqslant \int_a^b \big|G(t,s) - G(r,s)\big| \alpha(s) f\big(s, x(s)\big)\, ds \\
&\quad = \int_{A_n} \big|G(t,s) - G(r,s)\big| \alpha(s) f\big(s, x(s)\big)\, ds + \int_{[a_n,b_n]} \big|G(t,s) - G(r,s)\big| \alpha(s) f\big(s, x(s)\big)\, ds
\end{aligned}
$$

$$\leqslant 2\int\limits_{A_n} g(s)\alpha(s)f\big(s,x(s)\big)\,ds$$
$$+\int\limits_{[a_n,b_n]} \frac{\varepsilon\alpha(s)f(s,x(s))}{2(b-a)[1+\max_{w\in[a_n,b_n]}\alpha(w)\cdot\max_{(v,w)\in[a_n,b_n]\times[cD_n,d]}f(v,w)]}\,ds$$
$$<\varepsilon$$

for $(x,t,r)\in D\times[a,b]\times[a,b]$ with $|t-r|<\varepsilon$, that is, $\{Tx\colon x\in P_{c,d}\}$ is equicontinuous on $[a,b]$.

Let $\{x_n\}_{n\geqslant 1}$ be an sequence in $P_{c,d}$ with $\lim_{n\to\infty}x_n=x\in P_{c,d}$. It follows from ($C_1$) that for each $\varepsilon>0$ there exists some positive integer m_0 satisfying

$$\sup_{x\in P_{c,d}}\int\limits_{A_{m_0}} g(s)\alpha(s)f\big(s,x(s)\big)\,ds<\frac{\varepsilon}{4}. \tag{3.9}$$

Since f is continuous on $[a_{m_0},b_{m_0}]\times[cD_{m_0},d]$, there exists some positive constant δ satisfying

$$\big|f(v,t)-f(v,r)\big|<\frac{\varepsilon}{2\int_a^b g(s)\alpha(s)\,ds} \tag{3.10}$$

for $v\in[a_{m_0},b_{m_0}]$, $t,r\in[cD_{m_0},d]$ with $|t-r|<\delta$. Because $\lim_{n\to\infty}x_n=x\in P_{c,d}$, we can choose a positive integer $N>m_0$ such that

$$\|x_n-x\|<\delta,\quad n>N. \tag{3.11}$$

It follows from Lemma 2.1 that for each $n\geqslant 1$

$$cD_{m_0}\leqslant\min\big\{x_n(t),x(t)\big\}\leqslant\max\big\{x_n(t),x(t)\big\}\leqslant d,\quad t\in[a_{m_0},b_{m_0}]. \tag{3.12}$$

In light of (3.9)–(3.12), we deduce that for any $n>N$

$$\|Tx_n-Tx\|=\sup_{t\in[a,b]}\left|\int_a^b G(t,s)\alpha(s)\big[f\big(s,x_n(s)\big)-f\big(s,x(s)\big)\big]\,ds\right|$$
$$\leqslant\int_a^b g(s)\alpha(s)\big|f\big(s,x_n(s)\big)-f\big(s,x(s)\big)\big|\,ds$$
$$\leqslant\int\limits_{A_{m_0}} g(s)\alpha(s)\big|f\big(s,x_n(s)\big)-f\big(s,x(s)\big)\big|\,ds$$
$$+\int\limits_{[a_{m_0},b_{m_0}]} g(s)\alpha(s)\big|f\big(s,x_n(s)\big)-f\big(s,x(s)\big)\big|\,ds$$
$$\leqslant 2\sup_{u\in P_{c,d}}\int\limits_{A_{m_0}} g(s)\alpha(s)f\big(s,u(s)\big)\,ds$$
$$+\frac{\varepsilon}{2\int_a^b g(s)\alpha(s)ds}\int\limits_{[a_{m_0},b_{m_0}]} g(s)\alpha(s)\,ds$$
$$<\varepsilon,$$

which yields that $\lim_{n\to\infty} Tx_n = Tx$. That is, T is continuous on $P_{c,d}$. It follows from Arzela–Ascoli theorem that $T : P_{c,d} \to P$ is completely continuous.

Let x be in ∂P_j. It follows from Lemmas 2.1 and 2.2 that

$$\|x\| = j \geqslant x(t) \geqslant \|x\| l(t) \geqslant j \min\bigl\{l(p), l(q)\bigr\} \geqslant 2j \min\bigl\{h(p), h(q)\bigr\}, \quad t \in [p,q]. \tag{3.13}$$

In view of (3.1) and (3.13), we get that $f(t, x(t)) \geqslant jm$ for each $t \in [p,q]$. By virtue of Lemma 2.2, (C_1) and (3.3), we derive that

$$\begin{aligned}\|Tx\| &= \sup_{t\in[a,b]} \int_a^b G(t,s)\alpha(s) f\bigl(s, x(s)\bigr)\,ds \geqslant \int_p^q h(t) g(s)\alpha(s) f\bigl(s, x(s)\bigr)\,ds \\ &\geqslant \min\bigl\{h(p), h(q)\bigr\} jm \int_p^q g(s)\alpha(s)\,ds = j,\end{aligned}$$

which gives that

$$\|Tx\| \geqslant \|x\|, \quad x \in \partial P_j. \tag{3.14}$$

Put

$$B(x) = \bigl\{t \in [a,b]:\ x(t) > \beta\bigr\}, \quad \bar{x}(t) = \min\bigl\{x(t), \beta\bigr\}, \quad x \in \partial P_i.$$

It is clear that (3.4) yields that $M(\beta) < +\infty$ and hence $i < +\infty$. Let $x \in \partial P_i$. Obviously, $\bar{x} \in C[a,b]$. Note that $0 \leqslant x(t) \leqslant \|x\| = i$ for each $t \in [a,b]$ and there exists some $t_0 \in [a,b]$ satisfying $x(t_0) = \|x\|$. Consequently, we deduce that $0 \leqslant \bar{x}(t) = \min\{x(t), \beta\} \leqslant \min\{i, \beta\} = \beta$ for each $t \in [a,b]$ and $\bar{x}(t_0) = \min\{i, \beta\} = \beta$. Hence $\|\bar{x}\| = \beta$. Thus Lemmas 2.1 and 2.2 imply that

$$\begin{aligned}\bar{x}(t) &= \min\bigl\{x(t), \beta\bigr\} \geqslant \min\bigl\{il(t), \beta\bigr\} \\ &\geqslant \min\bigl\{2i \min\bigl\{h(p), h(q)\bigr\}, \beta\bigr\} \geqslant \beta \geqslant h(t)\|x\|, \quad t \in [p,q],\end{aligned}$$

that is, $\bar{x} \in \partial P_\beta$. For any $t \in B(x)$, we conclude easily that $\beta < x(t) \leqslant \|x\| = i$ and $f(t, x(t)) \leqslant rx(t) \leqslant ri$ by (3.2). It follows from (3.4) that

$$\begin{aligned}\|Tx\| &= \sup_{t\in[a,b]} \int_a^b G(t,s)\alpha(s) f\bigl(s, x(s)\bigr)\,ds \leqslant \int_a^b g(s)\alpha(s) f\bigl(s, x(s)\bigr)\,ds \\ &= \int_{B(x)} g(s)\alpha(s) f\bigl(s, x(s)\bigr)\,ds + \int_{[a,b]\setminus B(x)} g(s)\alpha(s) f\bigl(s, x(s)\bigr)\,ds \\ &\leqslant ri \int_{B(x)} g(s)\alpha(s)\,ds + \int_{[a,b]\setminus B(x)} g(s)\alpha(s) f\bigl(s, \bar{x}(s)\bigr)\,ds \\ &\leqslant ri \int_a^b g(s)\alpha(s)\,ds + \int_a^b g(s)\alpha(s) f\bigl(s, \bar{x}(s)\bigr)\,ds \leqslant rik^{-1} + M(\beta) \leqslant i,\end{aligned}$$

which gives that

$$\|Tx\| \leqslant \|x\|, \quad x \in \partial P_i. \tag{3.15}$$

Lemma 2.3, (3.14) and (3.15) guarantee that the operator T has at least one fixed point $x \in P_{j,i}$, which is a solution of Eqs. (1.4), (1.5). It follows from Lemmas 2.1 and 2.2 that

$$x(t) \geqslant \|x\| l(t) \geqslant 2jh(t) > 0, \quad t \in (a,b),$$

which implies that x is a positive solution of Eqs. (1.4), (1.5). This completes the proof. □

Theorem 3.2. *Let the function* $f:[a,b]\times(0,+\infty)\to[0,+\infty)$ *satisfy the following conditions*:

$$\lim_{s\to 0^+} \min_{t\in[p,q]} \frac{f(t,s)}{s} = +\infty \tag{3.16}$$

and

$$\limsup_{s\to+\infty} \max_{t\in[a,b]} \frac{f(t,s)}{s} < k. \tag{3.17}$$

If (C_1) *and* (C_2) *hold, then Eqs.* (1.4), (1.5) *possess a positive solution in* P.

Proof. Note that (3.16) and (3.17) imply that there exist positive constants β, j and r with $\beta > j$ and $r < k$ satisfying (3.1) and (3.2). Thus Theorem 3.2 follows from Theorem 3.1. This completes the proof. □

Theorem 3.3. *Let* (3.17), (C_1) *and* (C_2) *be satisfied. If one of the following conditions*:

$$\liminf_{s\to 0^+} \min_{t\in[p,q]} f(t,s) > 0 \tag{3.18}$$

and

$$\lim_{s\to 0^+} \min_{t\in[p,q]} f(t,s) = +\infty \tag{3.19}$$

holds, then Eqs. (1.4), (1.5) *possess a positive solution in* P.

Proof. Because each of (3.18) and (3.19) implies that (3.16) holds. Thus Theorem 3.3 follows from Theorem 3.2. This completes the proof. □

4. Examples

In this section, we construct two examples to illustrate the usefulness of the results obtained in Section 3.

Example 4.1. Consider the singular third order two-point boundary value problem:

$$x'''(t) - \frac{1}{t\sqrt{3-t}}\left[\frac{1}{40\sqrt{3}}t^2x(t) + \frac{1}{\sqrt[3]{x(t)}} + \sqrt{tx(t)}\,\bigl|\ln x(t)\bigr|\right] = 0, \quad 0 < t < 3, \tag{4.1}$$

$$x(0) = x(3) = x''(3) = 0. \tag{4.2}$$

Let $a = 0$, $b = 3$, $p = 1$, $q = 2$, $A_n = [0, a_n] \cup [b_n, 3]$, where $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ are strictly decreasing and strictly increasing sequences, respectively, with $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 3$

and $a_1 < 1.5 < b_1$. Assume that the functions $\alpha:(0,3)\to[0,+\infty)$, $g:[a,b]\to[0,+\infty)$ and $f:[0,3]\times(0,+\infty)\to[0,+\infty)$ are defined by

$$\alpha(t)=\frac{1}{t\sqrt{3-t}},\quad t\in(0,3),\qquad g(t)=t^2,\quad t\in[0,3],$$
$$f(t,s)=\frac{1}{40\sqrt{3}}t^2s+\frac{1}{\sqrt[3]{s}}+\sqrt{ts}|\ln s|,\quad (t,s)\in[0,3]\times(0,+\infty).$$

It is not difficult to verify that

$$\int_0^3 g(t)\alpha(t)\,dt=4\sqrt{3}=k^{-1}<+\infty,$$
$$\int_1^2 g(t)\alpha(t)\,dt=3^{-1}(14\sqrt{2}-16)>0,$$
$$\lim_{s\to0^+}\min_{t\in[1,2]}\frac{f(t,s)}{s}=+\infty,\qquad \lim_{s\to+\infty}\max_{t\in[0,3]}\frac{f(t,s)}{s}=0.9k<k.$$

It follows from Lemma 2.1 that for any $\rho>\delta>0$

$$\begin{aligned}
0&\leqslant\sup_{x\in P_{\delta,\rho}}\int_{A_n} g(t)\alpha(t)f\bigl(t,x(t)\bigr)\,dt\\
&=\sup_{x\in P_{\delta,\rho}}\int_{A_n}\frac{1}{t\sqrt{3-t}}\left[\frac{1}{40\sqrt{3}}t^2x(t)+\frac{1}{\sqrt[3]{x(t)}}+\sqrt{tx(t)}\bigl|\ln x(t)\bigr|\right]dt\\
&\leqslant\int_{A_n}\frac{1}{t\sqrt{3-t}}\left[\frac{1}{40\sqrt{3}}t^2\rho+\frac{3}{\sqrt[3]{\delta t(3-t)}}\right.\\
&\qquad\left.+\sqrt{\rho t}\bigl(|\ln\rho|+|\ln\delta|+3\ln 3+\bigl|\ln(3-t)\bigr|+|\ln t|\bigr)\right]dt\\
&\leqslant L\int_{A_n}\left[\frac{1}{\sqrt{3-t}}+\frac{1}{(3-t)^{5/6}}+\frac{|\ln(3-t)|}{\sqrt{3-t}}+\frac{t^{3/2}|\ln t|}{\sqrt{3-t}}\right]dt\\
&\to 0\quad\text{as } n\to+\infty,
\end{aligned}$$

where $L=27\rho+9\delta^{-1/3}+9\rho^{1/2}(|\ln\rho|+|\ln\delta|+3\ln 3)$. Consequently, Theorem 3.2 guarantees that Eqs. (4.1), (4.2) have a positive solution $x\in P$.

Example 4.2. Consider the singular third order two-point boundary value problem

$$\begin{aligned}
x'''(t)&-\frac{1}{(t-a)\sqrt{b-t}}\left[\frac{3}{4(2b+a)\sqrt{b-a}}\bigl(\sqrt{1+x^2(t)}-1\bigr)\left(\frac{t-a}{b-a}\right)^2\right.\\
&\left.+\left(2+\sin\frac{1}{x(t)}\right)\sqrt[3]{\frac{b-t}{b-a}}\right]=0,\quad a<t<b,
\end{aligned}\tag{4.3}$$

with the boundary condition (1.5). Define the functions $\alpha:(a,b)\to[0,+\infty)$, $g:[a,b]\to[0,+\infty)$ and $f:[a,b]\times(0,+\infty)\to[0,+\infty)$ by

$$\alpha(t)=\frac{1}{(t-a)\sqrt{b-t}},\quad t\in(a,b),\qquad g(t)=(t-a)^2,\quad t\in[a,b],$$

$$\begin{aligned}f(t,s)&=\frac{3}{4(2b+a)\sqrt{b-a}}\bigl(\sqrt{1+s^2}-1\bigr)\left(\frac{t-a}{b-a}\right)^2\\&\quad+\left(2+\sin\frac{1}{s}\right)\sqrt[3]{\frac{b-t}{b-a}},\quad (t,s)\in[a,b]\times(0,+\infty).\end{aligned}$$

Let $\{a_n\}_{n\geqslant1}$ and $\{b_n\}_{n\geqslant1}$ be strictly decreasing and strictly increasing sequences, respectively, with $\lim_{n\to\infty}a_n=a$, $\lim_{n\to\infty}b_n=b$ and $a_1<\frac{a+b}{2}<b_1$. Let p and q be constants with $a<p<q<b$. It is clear that

$$k^{-1}=\int_a^b g(t)\alpha(t)\,dt=2\bigl[b(b-a)^{1/2}-3^{-1}(b-a)^{3/2}\bigr]<+\infty,$$

$$\int_p^q g(t)\alpha(t)\,dt=2\bigl[\bigl(b(b-p)^{1/2}-3^{-1}(b-p)^{3/2}\bigr)-\bigl(b(b-q)^{1/2}-3^{-1}(b-q)^{3/2}\bigr)\bigr]>0,$$

$$\begin{aligned}\limsup_{t\to+\infty}\max_{t\in[a,b]}\frac{f(t,s)}{s}&\leqslant\limsup_{s\to+\infty}\left[\frac{k}{2}\bigl(\sqrt{1+s^2}-1\bigr)\left(\frac{t-a}{b-a}\right)^2+\left(2+\sin\frac{1}{s}\right)\sqrt[3]{\frac{b-t}{b-a}}\right]\\&=\frac{k}{2}<k,\end{aligned}$$

$$\liminf_{s\to0^+}\min_{t\in[p,q]}f(t,s)\geqslant\sqrt[3]{\frac{b-q}{b-a}}>0$$

and for any $d>c>0$

$$\begin{aligned}0&\leqslant\sup_{x\in P_{c,d}}\int_{A_n}g(t)\alpha(t)f\bigl(t,x(t)\bigr)\,dt\\&\leqslant\sup_{x\in P_{c,d}}\int_{A_n}\frac{t-a}{\sqrt{b-t}}\left[\frac{k}{2}\bigl(\sqrt{1+x^2(t)}-1\bigr)\left(\frac{t-a}{b-a}\right)^2+\left(2+\sin\frac{1}{x(t)}\right)\sqrt[3]{\frac{b-t}{b-a}}\right]dt\\&\leqslant\int_{A_n}\frac{t-a}{\sqrt{b-t}}\left[\frac{k}{2}\bigl(\sqrt{1+d^2}-1\bigr)\left(\frac{t-a}{b-a}\right)^2+3\sqrt[3]{\frac{b-t}{b-a}}\right]dt\\&\leqslant\left[\frac{k}{2}\bigl(\sqrt{1+d^2}-1\bigr)+3\right](b-a)\int_{A_n}\frac{1}{\sqrt{b-t}}\,dt\to0\quad\text{as }n\to+\infty.\end{aligned}$$

It follows from Theorem 3.3 that Eqs. (4.3), (1.5) possess a positive solution $x\in P$.

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